

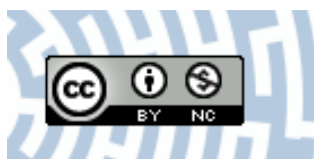


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**Title:** On the equivalence of pre-Schroder equations

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**Citation style:** Kalinowski Józef. (2007). On the equivalence of pre-Schroder equations. "Rocznik Akademii Górniczo-Hutniczej im. Stanisława Staszica, Opuscula Mathematica" (Vol. 27, nr 2 (2007), s. 253-257).



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## ON THE EQUIVALENCE OF PRE-SCHRÖDER EQUATIONS

**Abstract.** In the paper the equivalence of the system of two pre-Schröder functional equations (equations  $(S_n)$ ,  $(S_m)$  for  $m > n \geq 3$ ,  $n, m \in \mathbb{N}$ ) and the whole system  $(S)$ , is considered. The results solve the problem of Gy. Targonski [4] in a particular case.

**Keywords:** pre-Schröder equations, Targonski's problem, torsion free semigroups.

**Mathematics Subject Classification:** 39B62, 39B42.

### 1. INTRODUCTION

Let  $X$  be a set, and let  $g : X \rightarrow X$  be a given function. Then the equation

$$f(g(x)) = s \cdot f(x), \quad x \in X,$$

for the eigenfunction  $f : X \rightarrow Y$ , of the operator of substitution  $f \rightarrow f \circ g$ , where  $(Y, \cdot)$  is a commutative semigroup, corresponding to the eigenvalue  $s \in Y$ , is named the Schröder equation.

Iterating the Schröder equation  $n$  times, we obtain

$$f(g_n(x)) = s^n \cdot f(x) \quad x \in X,$$

where  $g_n$  denotes the  $n$ -th iterate of the function  $g$  for an integer  $n \geq 0$ , i.e.,

$$g_0(x) = x, \quad g_{n+1}(x) = g(g_n(x)), \quad x \in X.$$

Next, we raise both sides of the Schröder equation to the  $n$ -th power, getting

$$f^n(g(x)) = s^n \cdot f^n(x), \quad x \in X,$$

Eliminating the factor  $s^n$  from the above equations we arrive at the system

$$f^n(g(x)) = f(g_n(x)) \cdot f^{n-1}(x) \quad \text{for all integers } n \geq 2, \quad (S)$$

(for  $n = 1$  system  $(S)$  is not interesting, since it becomes an identity).

This infinite system (S) of functional equations has been introduced by Gy. Targonski under the name of the pre-Schröder system. The  $n$ -th equation of system (S) will be denoted by  $(S_n)$ .

If  $f$  fulfils the Schröder equation, then  $f$  also satisfies infinite system (S).

The problem of equivalence between the pre-Schröder equations was posed in 1970 by Gy. Targonski [4]. He also posed the question whether if a part of system (S) and whole system (S) are equivalent.

The positive solution of the problem was given in 1970 by Z. Moszner [3]. He proved that equation  $(S_2)$  and whole system (S) are equivalent under the assumption that  $Y$  is a countable set. In 1972 Gy. Targonski [5] proved the equivalence of  $(S_2)$  and (S) in the case where  $(Y, \cdot)$  is a commutative group.

The equivalence of (S) and of particular equations  $(S_n)$ ,  $n \geq 2$ , has been investigated in 1975 by J. Drewniak, J. Kalinowski [1] (see also chapter 9.2 in the book by M. Kuczma, B. Choczewski, R. Ger [2]).

The paper is a continuation of that research. We will consider the question of when the system of two equations  $(S_m)$ ,  $(S_n)$ ,  $m, n \in \mathbb{N}$ ,  $m > n \geq 3$ , and the whole system of the pre-Schröder equations (S) are equivalent.

## 2. PRELIMINARIES

Let  $\mathbb{N}$  denote the set of positive integers. We put  $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ .

Let  $(Y, \cdot)$  be a commutative semigroup. We denote by 0 a zero element in  $Y$ , such that

$$\bigwedge_{y \in Y} 0 \cdot y = 0,$$

provided that it does exist. It is obvious, that if a zero exists, it is unique.

In [1] we have proved the following

**Theorem 1.** *Let  $(Y, \cdot)$  be a commutative semigroup satisfying the following cancellation law*

$$\bigwedge_{x, y, z \in Y} (xy = xz \wedge x \neq 0) \implies (y = z). \quad (1)$$

*Then every function  $f$  verifying equation  $(S_2)$  satisfies all of the equations of system (S).*

We also proved [1] that no equation  $(S_n)$  for  $n \geq 3$  is equivalent to whole system (S). No two equations  $(S_m)$ ,  $(S_n)$ ,  $m \neq n$  for  $m, n \geq 3$ ,  $m, n \in \mathbb{N}$ , are equivalent, either.

## 3. MAIN RESULTS

In the paper, we suppose that for a fixed number  $n > 1$ ,  $n \in \mathbb{N}$ , a semigroup  $(Y, \cdot)$  is without the  $n$ -th degree torsion, i.e.,

$$\bigwedge_{x,y \in Y} (x^n = y^n) \implies (x = y). \quad (2)$$

**Theorem 2.** *Let  $(Y, \cdot)$  be a commutative semigroup satisfying cancellation law (1) and without the  $n$ -th degree torsion. If the function  $f$  satisfies system of two equations  $(S_n)$ ,  $(S_{n+1})$  for  $n \geq 3$ ,  $n \in \mathbb{N}$ , then  $f$  is a solution of whole system  $(S)$ .*

*Proof.* Let  $n \geq 3$ . Multiplying, by  $f^n(x)$ , both sides of equation  $(S_n)$  with variable  $x$  replaced by  $g(x)$  and using the commutativity of the multiplication, we obtain

$$f^n(g_2(x)) \cdot f^n(x) = f(g_{n+1}(x)) \cdot f^n(x) \cdot f^{n-1}(g(x)). \quad (3)$$

From (3), using the equation  $(S_{n+1})$ , we obtain

$$\begin{aligned} [f(g_2(x)) \cdot f(x)]^n &= [f(g_{n+1}(x)) \cdot f^n(x)] \cdot f^{n-1}(g(x)) = \\ &= f^{n+1}(g(x)) \cdot f^{n-1}(g(x)) = [f^2(g(x))]^n. \end{aligned}$$

Since  $(Y, \cdot)$  is a semigroup without  $n$ -th degree torsion, the function  $f$  satisfies equation  $(S_2)$ . By Theorem 1, we obtain the statement of the theorem.  $\square$

**Theorem 3.** *Let  $(Y, \cdot)$  be a commutative semigroup satisfying cancellation law (1) and without the  $n$ -th degree torsion. If the function  $f$  satisfies the system of two equations  $(S_n)$ ,  $(S_{2n})$  for  $n \geq 3$ ,  $n \in \mathbb{N}$ , then  $f$  is a solution of whole system  $(S)$ .*

*Proof.* Let  $n \geq 3$ . Multiplying both sides of equation  $(S_{2n})$  by  $f^{n-1}(g_n(x))$  and using equation  $(S_n)$  with variable  $x$  replaced by  $g_n(x)$ , we obtain

$$f^{2n}(g(x)) \cdot f^{n-1}(g_n(x)) = f^n(g_{n+1}(x)) \cdot f^{2n-1}(x). \quad (4)$$

Multiplying both sides of equation (4) by  $f^{n(n-1)}(g(x))$  and using equation  $(S_n)$  with variable  $x$  replaced by  $g(x)$ , we obtain

$$\begin{aligned} f^{n(n+1)}(g(x)) \cdot f^{n-1}(g_n(x)) &= f^n(g_{n+1}(x)) \cdot f^{n(n-1)}(g(x)) \cdot f^{2n-1}(x) = \\ &= [f(g_{n+1}(x)) \cdot f^{n-1}(g(x))]^n \cdot f^{2n-1}(x) = \\ &= f^{n^2}(g_2(x)) \cdot f^{2n-1}(x). \end{aligned} \quad (5)$$

From (5) and the equation  $(S_n)$ , there follows

$$\begin{aligned} [f(g_2(x)) \cdot f(x)]^{n^2} &= f^{n^2}(g_2(x)) \cdot f^{2n-1}(x) \cdot f^{(n-1)^2}(x) = \\ &= f^{n(n+1)}(g(x)) \cdot f^{n-1}(g_n(x)) \cdot f^{(n-1)^2}(x) = \\ &= f^{n(n+1)}(g(x)) \cdot [f(g_n(x)) \cdot f^{n-1}(x)]^{n-1} = \\ &= f^{n(n+1)}(g(x)) \cdot [f^n(g(x))]^{n-1} = [f^2(g(x))]^{n^2}. \end{aligned}$$

Because  $(Y, \cdot)$  is a semigroup without the  $n$ -th degree torsion, the function  $f$  verifies equation  $(S_2)$ . By Theorem 1,  $f$  is a solution of  $(S)$ . This completes the proof.  $\square$

**Remark 1.** Using the methods analogous to those in the proof of Theorem 3, we can prove that a solution of system  $(S_4)$ ,  $(S_6)$  is a solution of equation  $(S_2)$ . By Theorem 1, we obtain that system  $(S_4)$ ,  $(S_6)$  is equivalent to whole system  $(S)$ .

**Theorem 4.** Let  $(Y, \cdot)$  be a commutative semigroup satisfying cancellation law (1) and without the  $n$ -th degree torsion. If the function  $f$  satisfies the system of two equations  $(S_{n+1})$ ,  $(S_{2n+1})$  for  $n \geq 2$ ,  $n \in \mathbb{N}$ , then  $f$  is a solution of whole system  $(S)$ .

*Proof.* Let  $n \geq 2$ . Multiplying both sides of equation  $(S_{2n+1})$  by  $f^n(g_n(x))$  and using equation  $(S_{n+1})$  with variable  $x$  replaced by  $g_n(x)$ , we obtain

$$f^{2n+1}(g(x)) \cdot f^n(g_n(x)) = [f(g_{2n+1}(x)) \cdot f^n(g_n(x))] \cdot f^{2n}(x) = f^{n+1}(g_{n+1}(x)) \cdot f^{2n}(x).$$

Multiplying both sides of the above equation by  $f^{n(n-1)}(x)$  and using the commutativity of the multiplication and equation  $(S_{n+1})$ , we obtain

$$\begin{aligned} f^{2n+1}(g(x)) \cdot f^n(g_n(x)) \cdot f^{n(n-1)}(x) &= f^{n+1}(g_{n+1}(x)) \cdot f^{n(n+1)}(x) = \\ &= [f(g_{n+1}(x)) \cdot f^n(x)]^{n+1} = [f^{n+1}(g(x))]^{n+1} = f^{(n+1)^2}(g(x)). \end{aligned}$$

Therefore

$$f^{2n+1}(g(x)) \cdot f^n(g_n(x)) \cdot f^{n(n-1)}(x) = f^{(n+1)^2}(g(x)). \quad (6)$$

Let us consider two possible cases:

(a)  $f^{2n+1}(g(x)) \neq 0$ .

Because  $(Y, \cdot)$  satisfies cancellation law (1), from equation (6) we obtain

$$f^n(g_n(x)) \cdot f^{n(n-1)}(x) = f^{n^2}(g(x)). \quad (7)$$

(b)  $f^{2n+1}(g(x)) = 0$ .

Since  $(Y, \cdot)$  is a semigroup satisfying cancellation law (1), then  $Y$  has no zero divisors and we obtain  $f(g(x)) = 0$ . Replacing the variable  $x$  by  $g(x)$  in equation  $(S_{n+1})$ , we obtain

$$f^{n+1}(g_2(x)) = f(g_{n+2}(x)) \cdot f^n(g(x)),$$

whence  $f(g_2(x)) = 0$ . Replacing  $x$  by  $g_2(x)$  in equation  $(S_{n+1})$ , we obtain  $f(g_3(x)) = 0$ . By induction, there is  $f(g_n(x)) = 0$  for  $n \in \mathbb{N}$ . Then

$$f^n(g_n(x)) \cdot f^{n(n-1)}(x) = 0$$

and (b) yields  $f^{n^2}(g(x)) = 0$ . So in case (b), equation (7) is satisfied too.

The equation (7) can be written in the form

$$[f(g_n(x)) \cdot f^{n-1}(x)]^n = [f^n(g(x))]^n.$$

Owing to (2), the function  $f$  satisfies equation  $(S_n)$ . Then the function  $f$  fulfils the system of equations  $(S_n)$ ,  $(S_{n+1})$ . By Theorem 2, the function  $f$  satisfies whole system  $(S)$ .  $\square$

**Remark 2.** Using the methods analogous to those in the proof of Theorem 4, we can prove that  $(S_4)$ ,  $(S_7)$  imply  $(S_3)$ , as well as that  $(S_4)$ ,  $(S_9)$  imply  $(S_3)$ . Then, by Theorem 2, we obtain that both systems  $(S_4)$ ,  $(S_7)$  and  $(S_4)$ ,  $(S_9)$  are equivalent to whole system  $(S)$ .

## REFERENCES

- [1] J. Drewniak, J. Kalinowski, *Les relations entre les équations pré-Schröder I*, Ann. Polon. Math. **32** (1976), 5–11.
- [2] M. Kuczma, B. Choczewski, R. Ger, *Iterative Functional Equations*, Encyclopedia of Mathematics and its Applications, Vol. 32, Cambridge Univ. Press, Cambridge-New York-New Rochelle-Melburne-Sydney 1990.
- [3] Z. Moszner, *P 63 S 1*, *P 63 S 2*, Aequationes Math. **4** (1970), 395.
- [4] Gy. Targonski, *Problem P 63*, ibidem **4** (1970), 251.
- [5] Gy. Targonski, *On pre-Schröder equations*, ibidem **8** (1972), 157.

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*Received: December 6, 2004.*